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A Time-Symmetric Classical Electrodynamics

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Abstract

In this theory, both the advanced and retarded Liénard-Wiechert potentials are used to compute the fields of a charged point particle. The incoming radiation from the advanced fields balances the outgoing radiation of the retarded fields, and we assume that there are no radiation reaction terms in the equations of motion of the particles. We further assume that only retarded fields act on particles through the Lorentz force, and that advanced fields act on antiparticles. This is a theory that is symmetric under time reflection (reversal of the direction of motion plus charge conjugation).

1. *Introduction*

The most successful theory of elementary particles is, to date, quantum electrodynamics. The agreement between experimental results and calculations is overwhelming in a number of applications. Nevertheless, those calculations are carried out within the framework of quantum field theory and perturbation expansion, where infinite renormalizations are necessary to avoid well-known problems. Thus, in spite of its success, there still is considerable doubt about the possibility of putting these calculations on a sound mathematical basis.

We have explored the alternative approach of relativistic quantum mechanics (Marx, 1969, 1970a, 1970c), and we have been able to make considerable progress. We have used causal Green functions in a many-times formalism to allow for pair creation and annihilation, and conservation of charge for particles in an external electromagnetic field leads to a probabilistic interpretation. The number of particles is constant, but they can turn around in time; the formalism of quantum field theory can then be used much in the manner of non-relativistic many-particle theory (Marx, 1972). But we have had great difficulty in even writing down equations for interacting charged particles with a dynamical electromagnetic field, although we have explored a possible approach in non-relativistic quantum mechanics (Marx, 1974). One reason for this problem is that there is still considerable ambiguity in the interaction of classical charged particles with a classical electromagnetic field. The most satisfactory approach, as presented by Rohrlich (1965), is not very helpful in the context of quantum mechanics. Although the equations can be derived from an action principle, they do not lend themselves to a Hamiltonian formulation. The appearance of third derivatives with respect to time or proper time, the asymptotic conditions and preacceleration are bothersome even within a classical context. Also the interaction of classical fields with the electromagnetic field is not free of questions of gauge invariance and Lorentz covariance, especially when boundary conditions are taken into account (Marx, 1970b).

We recently proposed (Marx, 1975a) a different way of looking at a particle with its Coulomb field, by integrating the stress-energy tensor over the future light cone. When the fields are referred to the vertex of the light cone, the integral of the tensor over a tube around the world line contains no third derivative term. We thus were led to an equation of motion without the Schott term, but consistency required a variable mass, that decreased as the rest energy fed the radiation.

Here we present an alternative point of view (Marx, 1975b) that makes use of both advanced and retarded fields, particles and antiparticles in a time-symmetric way. The advanced Liénard-Wiechert potentials represent incoming radiation fields that are absorbed by a particle at the same rate as it emits energy through the retarded fields. Thus, there is no need for a radiation reaction term due to local conservation of energy and momentum around the world line of a particle. We assume that only retarded fields act on particles, avoiding any causality problems in macroscopic electrodynamics, where radiation reaction does not play a significant role. Antiparticles are affected by advanced fields only, which makes this theory symmetric under time reflection.

The main difficulties lie in the lack of a variational principle for the complete system, although a single-particle Lagrangian is readily available and, partly as a consequence, the lack of a global conservation law for particles and fields.

The electromagnetic fields give rise to absorption and emission of radiation, which travels with the speed of light, but fields are of secondary importance for the interaction between particles, which is that of an action-at-a-distance theory.

We use a time-favoring metric with a modified summation convention for repeated lower Greek indices. We use natural units so that $c = 1$, $\epsilon_0 = 1$, and $\mu_0 = 1$. We generally follow the conventions in Marx (1975a) and other references.

2. *Energy and Momentum of the Advanced Fields*

We first compute the stress-energy tensor for the advanced electromagnetic fields generated by a classical charged point particle. The world line of this particle is given by the parametric equations

$$x_\mu = \xi_\mu(\tau) \tag{2.1}$$

in terms of the proper time τ. The advanced fields at a point x are given by the corresponding Liénard-Wiechert potentials

$$A_\mu^A(x) = \frac{e}{4\pi} \frac{u_\mu[\tau_A(x)]}{\rho} \tag{2.2}$$

where u_μ is the four-velocity, the advanced proper time τ_A is determined by the forward light cone from x where it intersects the world line, and the invariant distance ρ is obtained from the null vector R by

$$\rho = -R \cdot u \tag{2.3}$$

This advanced null vector is given by

$$R_\mu^A(x) = x_\mu - \xi_\mu[\tau_A(x)] \tag{2.4}$$

where τ_A is determined by

$$R_\mu(x)R_\mu(x) = 0, \qquad R_0(x) < 0 \tag{2.5}$$

The spacelike unit four vector $b(x)$ normal to the world line at τ_A can be found from

$$R = \rho(b - u) \tag{2.6}$$

The fields and the stress-energy tensor are then computed in the usual manner. In terms of the quantities defined above and the four-acceleration w_μ, the symmetrized stress-energy tensor is

$$\begin{aligned}\Theta_{\mu\nu} = &-(e^2/16\pi^2\rho^4)\{[(1-\rho b\cdot w)^2 + \rho^2 w^2]b_\mu b_\nu + [\rho b\cdot w - \rho^2(b\cdot w)^2\\ &-\rho^2 w^2](b_\mu u_\nu + b_\nu u_\mu) - [1-\rho^2(b\cdot w)^2 - \rho^2 w^2]u_\mu u_\nu\\ &-\rho[w_\mu(b_\nu - u_\nu) + w_\nu(b_\mu - u_\mu)] + g_{\mu\nu}/2\}\end{aligned} \tag{2.7}$$

The sign of $\Theta_{\mu\nu}$ is such that the energy density Θ_{00} is positive.

We next compute the energy-momentum flux "across" the backward light cone. We choose the surface element so that there is continuity with the forward light cone (see Figure 1), and its equation is

$$d\sigma_\mu = \rho^2(u_\mu - b_\mu)d\rho d\Omega \tag{2.8}$$

We integrate over the light cone outside a sphere of radius ϵ, and we obtain the Coulomb energy-momentum

$$P_\mu{}^C = (e^2/8\pi\epsilon)u_\mu \tag{2.9}$$

The tube around the world line at a distance ϵ measured along the backward light cone is not exactly the same as the corresponding one on the forward light cone. Its equation is

$$x(\tau, \theta, \phi) = \xi(\tau) + \epsilon[b(\tau, \theta, \phi) - u(\tau)] \tag{2.10}$$

and the corresponding surface element is

$$d\sigma_\mu = \epsilon^2[(1 - \epsilon b\cdot w)b_\mu + \epsilon b\cdot w u_\mu]d\Omega d\tau \tag{2.11}$$

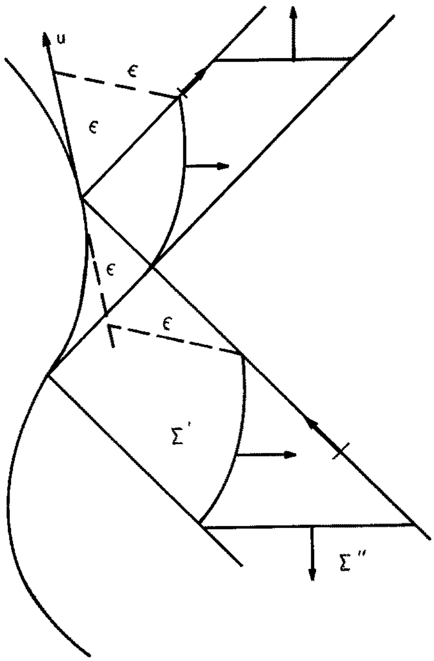

Figure 1—Advanced and retarded light cones and segments of the corresponding tubes around the world line of a particle.

Thus, we find

$$\Theta_{\mu\nu}d\sigma_\nu = -(e^2/16\pi^2\epsilon^2)\{\epsilon b\cdot w(-\tfrac{1}{2}+\epsilon b\cdot w)u_\mu + [-\tfrac{1}{2}+\tfrac{3}{2}\epsilon b\cdot w + \epsilon^2(b\cdot w)^2]b_\mu - \epsilon^2 w^2(b_\mu - u_\mu) + \epsilon w_\mu\}d\Omega d\tau \tag{2.12}$$

and we integrate over the solid angle to obtain

$$\int_{\Sigma'}\Theta_{\mu\nu}d\sigma_\nu = (e^2/4\pi\epsilon)\int_{\tau_1}^{\tau_2} d\tau(-\tfrac{1}{2}w_\mu - \tfrac{2}{3}\epsilon w^2 u_\mu) \tag{2.13}$$

The first term can be integrated and represents the change in the Coulomb energy, and the second one corresponds to incoming radiation. By the divergence theorem, the same rate of radiation is obtained from the integral over Σ'', when this surface recedes to infinity.

3. *Antiparticles*

We have seen elsewhere (Marx, 1970a, 1970c) that antiparticles in relativistic quantum mechanics are represenetd by solutions of wave equations that propagate backwards in time. For free fields, they correspond to the negative-frequency part of the solution and we have to specify it at the final time when we use a causal Green function or Feynman propagator.

Consequently, in the classical theory we choose the parameter on the world line of an antiparticle so that the proper time decreases with increasing coordinate time; that is, the four-velocity is a unit tangent vector pointing in the direction of the past. Thus, the signs of the velocity terms have to be changed throughout. The four-acceleration, on the other hand, remains unchanged.

If a time reversal operation is performed on the coordinate system (or if an observer made out of antimatter moves along the time axis in the negative direction), the advanced fields of the antiparticle correspond to emission of radiation into its future, the normal observer's past. This radiation then appears the same way to the antimatter observer as the usual radiation emitted by particles appears to the observer. The advanced fields, on the other hand, appear as incoming radiation converging to the particle or antiparticle where it is absorbed.

We can now define the energy-momentum vector either in the usual manner or with a change in sign. We set

$$p_\mu = -mu_\mu \tag{3.1}$$

for antiparticles, which makes the energy positive as far as the observer is concerned. The sign of the charge of an antiparticle is also to some extent arbitrary. Here we choose the charge with the opposite sign of the particle charge, although in relativistic quantum mechanics both particles and antiparticles are represented by the same field and only one charge appears. These ambiguities may disappear if a classical theory were to describe pair creation and annihilation, but we do not expect this to be the case.

When we describe particles and antiparticles in this manner, we have symmetry under time reflection (Wigner's reversal of the direction of motion plus charge conjugation).

4. *Particle Equations of Motion*

We have to look for an interaction between charged particles that does not contradict causality as found in macroscopic electromagnetic theory. This requirement is obviously satisfied when we assume that the Lorentz force acting on particles is due to the retarded field from all other particles. We furthermore assume that there is no radiation reaction force, and the equation for particle i is

$$m_i w_{i\mu} = -e_i F^R_{i\mu\nu}[\xi_i(\tau_i)] u_{i\nu}(\tau_i) \tag{4.1}$$

with no sum over i, and where the retarded field at the event ξ_i is obtained from the world lines of the other particles and antiparticles in the system. If the total number of world lines is N, we can write

$$F^R_{i\mu\nu}[\xi_i(\tau_i)] = \sum_{j=1}^{N}{}' F^R_{ij\mu\nu} \tag{4.2}$$

where the prime indicates that the term $j = i$ is excluded from the sum, and

$$F^R_{ij\mu\nu} = \frac{e_j}{4\pi}\left\{\frac{u_{j\mu}R_{ij\nu} - u_{j\nu}R_{ij\mu}}{(R_{ij}\cdot u_j)^3} - \left[\frac{w_{j\mu}R_{ij\nu} - w_{j\nu}R_{ij\mu}}{(R_{ij}\cdot u_j)^2} - \frac{(u_{j\mu}R_{ij\nu} - u_{j\nu}R_{ij\mu})R_{ij}\cdot w_j}{(R_{ij}\cdot u_j)^3}\right]\right\} \tag{4.3}$$

$$R_{ij} = \xi_i(\tau_i) - \xi_j[\tau_{jR}(\tau_i)] \tag{4.4}$$

with no sum over i or j. The proper time τ_{Rj} at which ξ_j, u_j, and w_j are evaluated is determined by the intersection of the jth world line and the backward light cone from ξ_i. This is then a retarded action-at-a-distance theory, and the fields serve only subsidiary roles in the equation of motion for the particles. We should note that the directions of the velocities in equation (4.3) do not affect the fields. There is one set of equations of motion for each particle, and the system is the usual set of difference-differential equations. Only three out of the four equations in each set are independent, and the fourth one relates the change in mechanical energy to the work done by the Lorentz force. In three-vector notation, equations (4.1) reduce to

$$d\mathbf{p}_i/dt_i = e_i(\mathbf{E}_i^R + \mathbf{v}_i \wedge \mathbf{B}_i^R) \tag{4.5}$$

$$dp_{0i}/dt_i = e_i\mathbf{E}_i^R \cdot \mathbf{v}_i \tag{4.6}$$

where

$$\mathbf{p}_i = m_i\mathbf{v}_i(1 - \mathbf{v}_i^2)^{-1/2} \tag{4.7}$$

$$p_{0i} = m_i(1 - \mathbf{v}_i^2)^{-1/2} \tag{4.8}$$

$$\mathbf{v}_i = d\xi_i/dt_i \tag{4.9}$$

A part of the terms in the sum in equation (4.2) can be grouped in an external electromagnetic field, which is given and comes from particles not explicitly included in the system.

There is no macroscopic restriction on the interaction between antiparticles or between particles and antiparticles. We thus can invoke the symmetry under time reflection found in Sections 2 and 3 to postulate that the advanced fields from particles and antiparticles act on an antiparticle. Thus, the retarded fields and retarded proper time in the above equation have to be replaced by the advanced ones when the ith world line corresponds to an antiparticle.

In addition to the equations of motion, we have to specify boundary conditions. As in nonrelativistic mechanics, we have to give two three-vectors for each particle and antiparticle. Following the leads from relativistic quantum mechanics, we would specify the position and velocity of particles at the initial time and those of antiparticles at the final time, and seek to determine the world lines within that interval. The boundaries of the space-time region can be generalized to parallel spacelike planes or more general spacelike surfaces. But, since the interactions are not instantaneous, we can limit ourselves to the region of interest only if the separations of particles and antiparticles are such that interactions are negligible and the world lines outside the region can be approximated by straight lines. Otherwise, complete world lines have to be determined. There is control over these conditions when we deal with particles or antiparticles only, but in the mixed case the dynamics cannot be separated from the restrictions on boundary conditions.

We have not found a variational approach to these equations of motion for the system as a whole. The equations of motion of a relativistic particle in an external electromagnetic field can be obtained, for instance, from the Lagrangian

$$L = -m(1 - \mathbf{v}^2)^{1/2} - e(A_0 - \mathbf{v} \cdot \mathbf{A}) \tag{4.10}$$

and the same can be used for any given particle or antiparticle if the appropriate Liénard-Wiechert potentials are used for all other particles and antiparticles, assuming their world lines are fixed. A single-particle Hamiltonian can be useful in a many-times formalism for relativistic quantum mechanics.

5. *Energy-Momentum Balance Considerations*

Conservation of energy and momentum follows from invariance of closed systems under space-time translations when the equations of motion or particles and fields can be derived from an action principle. But even in such a case, the meaning of the different terms that arise for a system of particles and a dynamical electromagnetic field (Rohrlich, 1965) is quite obscure.

In our present theory, we can consider separately the emission and absorption of radiation by a particle and the interaction between particles.

For each particle, the Coulomb energy as defined by the integral of the retarded and advanced stress-energy tensors over the light cones is included in the mass of the particles. The term corresponding to the flux of this energy across the tube around the world line can be integrated to the beginning and end of the world line, if any. Creation and annihilation of particles should not be expected to be accounted for in classical theories anyway. The particle emits radiation that flows with the speed of light and it absorbs the same amount, represented by the advanced fields propagating backwards in time. Thus, we have a balance of energy and momentum without the need to feed the radiation from a Schott term or a decreasing mass. We explicitly exclude any interaction energy between the advanced and retarded fields.

The situation is less clear as far as the energy and momentum of interaction are concerned. The electrostatic potential energy of two particles is a function of the simultaneous positions, a concept that is difficult to generalize to a relativistic treatment, when interactions are retarded. It is straightforward to compute the energy-momentum flux across a tube around the world line due to a mixed term in the stress-energy tensor,

$$\Theta_{i\mu\nu} = -F_{i\alpha\mu}F'_{i\alpha\nu} - F_{i\alpha\nu}F'_{i\alpha\mu} + \tfrac{1}{2}F_{i\alpha\beta}F'_{i\alpha\beta}g_{\mu\nu} \tag{5.1}$$

where $F_{i\mu\nu}$ is the advanced or retarded field at the tube around particle i due to all other particles and antiparticles, as given by equation (4.2), and $F'_{i\mu\nu}$ is the advanced or retarded field due to the ith particle. When the radius ϵ of the tube tends to zero, the value of the fields $F_{i\mu\nu}$ on the world line can be used, only the velocity term in $F_{i\mu\nu}$ contributes, and only the term with $\epsilon^2 b_\mu$ has to be kept for the surface element in equation (2.11). We find

$$P_\mu = -e_i \int_{\tau_1}^{\tau_2} F^R_{i\mu\alpha} u_{i\alpha}\, d\tau_i \tag{5.2}$$

for the retarded fields, which is equal to the change in energy and momentum of the particle.

In electrodynamics with retarded fields only, the change in energy and momentum of the field is equal and opposite to that of the particles, but in the present theory particles and antiparticles give rise to both retarded and advanced fields, while only the retarded field acts on the particle and only the advanced one on the antiparticle.

Even in the case of purely retarded interactions energy balance is difficult to interpret. When two particles of the same charge and mass are initially moving towards each other on a straight line, an analysis of the retarded equations of motion with no radiation reaction shows (Huschilt *et al.*, 1973) that after the collision the speed of the particles at the same distance is greater than before the collision. This is a consequence of the fact that the retarded distance between the particles is greater when the particles are approaching each other than when they are receding from each other, at the same position. Conservation of energy is maintained when the negative energy of the interaction radiation is taken into account. In other words, the kinetic energy gained by the particles is supplied by the electromagnetic radiation energy due to the particles. The authors conclude from this that the radiation reaction should not be dropped from the equations of motion, but we think that this explanation does not address the problem of conservation of energy.

Furthermore, the concept of the interaction radiation is obscure in terms of the particles that emit it. The radiation emitted by a single particle stays between two light cones and can be identified as having been emitted at a particular segment of the world line and moving out with the speed of light. On the other hand, no such interaction energy originates from given elements of two world lines, as the regions between the two sets of light cones intercepted by a spacelike plane shift as the plane recedes into the future.

We thus propose to limit the concept of emission and absorption of radiation to the fields from single particles, and to consider the interaction between particles as an action at a distance.

6. *Concluding Remarks*

We have proposed a way of formulating the electromagnetic interactions between classical charged particles that is symmetric under time reflection.

Antiparticles are not just particles with the opposite sign of the charge, but they are accelerated by advanced fields instead of retarded ones.

We have generalized the view of a charged particle in Marx (1975a) to include advanced fields. Thus, a charged particle includes the Coulomb fields on both the forward and backward light cones, it emits radiation at the usual rate when it is accelerated, and it also absorbs precisely the same amount of radiation Consequently, no radiation reaction term is needed in the equations of motion to balance the radiation.

The interaction between charged particles is of the action-at-a-distance type; retarded fields produce a Lorentz force on particles only, and advanced fields act on antiparticles. We do not have a conservation of total energy and momentum for the system, but we find that this concept is of dubious value in a relativistic theory if the balance includes contributions from the fields.

Since classical theories are of very limited value in their use for elementary particles, the real test of such a theory lies in its internal consistency and a reasonable correspondence with a successful quantum theory.

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